(1)

Solution of the Numerical Problems of Mid Term II

Signals And Systems EC-402

(4)

LTI system is define as

$$y[n] = y[n-1] + y[n-2] + y[n-3] + x[n-1]$$

$$\Rightarrow y[n] - y[n-1] - y[n-2] - y[n-3] = x[n-1]$$

Taking the z transform of above difference equation and using the delay property of the z transform

$$x[n] \longrightarrow X(z)$$

$$x[n-n_0] \longrightarrow z^{-n_0} X(z)$$

Now

$$\Rightarrow Y(z)[1 - z^{-1} - z^{-2} - z^{-3}] = z^{-1} X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = H(z) = \frac{z^{-1}}{1 - z^{-1} - z^{-2} - z^{-3}}$$

Since factors are not possible in the above equation we have to use the Long Division Method to calculate h(n) from H(z)

2

$$
\begin{array}{r|l}
1-z^{-1}-z^{-2}-z^{-3} & \cancel{z^{-1}} \quad (z^{-1}+z^{-2}+2z^{-3}+4z^{-4} \\
& -\cancel{z^{-1}} - z^{-2} - z^{-3} - z^{-4} \\
& \quad + \quad + \quad + \\
\hline
& \cancel{z^{-2}} + z^{-3} + z^{-4} \\
& -\cancel{z^{-2}} - z^{-3} - z^{-4} - z^{-5} \\
& \quad + \quad + \quad + \\
\hline
& \cancel{2z^{-3}} + 2z^{-4} + 2z^{-5} \\
& -\cancel{2z^{-3}} - 2z^{-4} - 2z^{-5} - 2z^{-6} \\
& \quad + \quad + \quad + \\
\hline
& 4z^{-4} + 3z^{-5} + 2z^{-6}
\end{array}
$$

$$H(z) = \{ z^{-1} + z^{-2} + 2z^{-3} + 4z^{-4} + \dots \}$$

$$h(n) = \{ \dots, 1, 1, 2, 4, \dots \}$$

Since

$$x(n) = (0.5)^n u(n)$$

$$X(z) = \left[\frac{1}{1-\frac{1}{2}z^{-1}} \right]$$

$$Y(z) = [H(z)\; X(z)]$$

$$= \left[\frac{z^{-1}}{1-z^{-1}-z^{-2}-z^{-3}} \right] \left[\frac{1}{1-\frac{1}{2}z^{-1}} \right]$$

$$= \left[\frac{2z^{-1}}{(1-z^{-1}-z^{-2}-z^{-3})(2-z^{-1})} \right]$$

$$= \left[\frac{2z^{-1}}{2-2z^{-1}-2z^{-2}-2z^{-3}+z^{-1}-z^{-2}-z^{-3}-z^{-4}} \right]$$

Using the long division method (3)

$$2-3z^{-1}-z^{-2}-z^{-3}+z^{-4}\,\overline{)\,2z^{-1}\,}\,(z^{-1}+3z^{-2}+5z^{-3}$$

$$\underline{\cancel{2z^{-1}} - 6z^{-2} - z^{-3} - z^{-4} + z^{-5}}$$

$$6z^{-2}+z^{-3}+z^{-4}+z^{-5}$$

$$\underline{\cancel{6z^{-2}} - 9z^{-3} - 3z^{-4} - 3z^{-5} + 3z^{-6}}$$

$$10z^{-3}+4z^{-4}+2z^{-5}-3z^{-6}$$

$$\underline{10z^{-3} - 15z^{-4} - 5z^{-5} - 5z^{-6} + 5z^{-7}}$$

$$19z^{-4}+7z^{-5}+2z^{-6}-5z^{-7}$$

$$Y(z) = z^{-1}+3z^{-2}+5z^{-3}+\frac{19}{2}z^{-4}+\ldots$$

$$y[n] = \{\ldots, 1, 3, 5, \tfrac{19}{2}, \ldots\}$$

(2)

$$\frac{dy(t)}{dt} + 3y(t) = 2x(t)$$

Taking the Fourier transform of the differential equation and using the property of the Fourier transforms

$$(j\omega)\,Y(j\omega) + 3\,Y(j\omega) = 2X(j\omega)$$

$$\Rightarrow \frac{Y(j\omega)}{X(j\omega)} = \left[\frac{2}{3+j\omega}\right]$$

4

$$\Rightarrow H(j\omega) = \left[\frac{2}{3+j\omega}\right]$$

Taking the inverse fourier transform

$$h(t) = \left[2e^{-3t}u(t)\right]$$

Now

$$x(t) = e^{-t}u(t)$$

$$X(j\omega) = \left[\frac{1}{1+j\omega}\right]$$

$$Y(j\omega) = \left[X(j\omega)H(j\omega)\right]$$

$$\Rightarrow Y(j\omega) = \left[\frac{2}{(3+j\omega)(1+j\omega)}\right]$$

$$\Rightarrow Y(j\omega) = \left[\frac{A}{3+j\omega} + \frac{B}{1+j\omega}\right]$$

$$A = \frac{2}{1+j\omega}\bigg|_{j\omega=-3} = \frac{2}{1-3}$$

$$= \frac{2}{-2} = -1$$

$$B = \frac{2}{3+j\omega} = \frac{2}{3-1} = \frac{2}{2} = 1$$

$$Y(j\omega) = \left[-\frac{1}{3+j\omega} + \frac{1}{1+j\omega}\right]$$

Taking the inverse fourier transform

$$\boxed{y(t) = \left[e^{-t} - e^{-3t}\right]u(t)}$$

(5) From the figure it is clear that (5)

$$x(t) = \begin{cases} 5 & 0<t<2 \\ 0 & 2<t<5 \end{cases}$$

and the time period of the signal is 5ms (Unit of the time is assume)

Fundamental Angular frequency

$$W_0 = \frac{2\pi}{T} = \frac{2\pi}{5} \text{ Rad/ms}$$

Now, we know that

$$c_n = \frac{1}{T}\int_0^T \left[x(t)\, e^{-jW_0tn}\right] dt$$

$$= \frac{1}{5}\int_0^5 \left[x(t)\, e^{-jnW_0t}\right] dt$$

$$= \frac{1}{5}\int_0^2 \left[x(t)\, e^{-jW_0nt}\right] dt$$

$$= \int_0^2 e^{-jW_0nt}\, dt$$

$$= -\frac{1}{jW_0n}\, e^{-jW_0nt}\Big|_0^2$$

$$= \left[\frac{1-e^{-2jW_0n}}{jnW_0}\right]$$

$$\Rightarrow c_n = j\left[\frac{e^{-2jW_0n}-1}{nW_0}\right]$$

$$c_0 = j\left[\frac{0}{0}\right] = \frac{0}{0}$$

Now Using the L HOSPITAL RULE (6)

$$c_0 = j\left[\frac{e^{-j2nw_0} - 1}{nw_0}\right]$$

$$c_0 = \lim_{n\to 0} \frac{j \times (-j2w_0)\, e^{-j2nw_0}}{w_0}$$

$$= \lim_{n\to 0} 2e^{-j2nw_0} = 2$$

$$|c_0| = 2 \qquad \angle c_0 = 0$$

Now again

$$c_n = j\left[\frac{e^{-2jnw_0} - 1}{nw_0}\right]$$

$$= \frac{j}{nw_0}\left[\cos 2nw_0 - j\sin 2nw_0 - 1\right]$$

$$= \frac{1}{nw_0}\left[-j^2 \sin 2nw_0 + j(\cos 2nw_0 - 1)\right]$$

$$c_n = \frac{1}{nw_0}\left[\sin 2nw_0 + j(\cos 2nw_0 - 1)\right]$$

$$|c_n| = \frac{1}{nw_0}\left[(\sin 2nw_0)^2 + (\cos 2nw_0 - 1)^2\right]^{1/2}$$

$$= \frac{1}{nw_0}\left[\sin^2 2nw_0 + \cos^2 2nw_0 + 1 - 2\cos 2nw_0\right]^{1/2}$$

$$\boxed{|c_n| = \frac{1}{nw_0}\left[2 - 2\cos 2nw_0\right]^{1/2}}$$

7

$$\angle c_n = \tan^{-1}\left|\frac{\cos 2n w_0 - 1}{\sin 2n w_0}\right|$$

$$|c_1| = \frac{1}{w_0}\left[2 - 2\cos 2n w_0\right]^{1/2}$$

$$= \frac{5}{2\pi}\left[2 - 2\cos\left(\frac{4\pi}{5}\right)\right]^{1/2}$$

$$|c_2| = \frac{5}{2\pi}\left[2 - 2\cos\left(\frac{8\pi}{5}\right)\right]^{1/2}$$

$$|c_3| = \frac{5}{2\pi}\left[2 - 2\cos\left(\frac{12\pi}{5}\right)\right]^{1/2}$$

$$\angle c_1 = \tan^{-1}\left|\frac{\cos(4\pi/5) - 1}{\sin(4\pi/5)}\right|$$

$$\angle c_2 = \tan^{-1}\left|\frac{\cos(8\pi/5) - 1}{\sin(8\pi/5)}\right|$$

$$\angle c_3 = \tan^{-1}\left|\frac{\cos(12\pi/5) - 1}{\sin(12\pi/5)}\right|$$

3 A causal LTI system is describe as

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n]$$

Now taking the DTFT of the above difference equation and using the ~~th~~ time delay property of the DTFT

$$y[n] \longrightarrow Y[e^{jw}]$$

$$y[n-n_0] \longrightarrow e^{-jn_0 w}\, Y[e^{jw}]$$

(8)

$$\Rightarrow Y[e^{jw}] - \frac{3}{4}e^{-jw}\,Y[e^{jw}] + \frac{1}{8}e^{-2jw}\,Y[e^{jw}] = X[e^{jw}]$$

$$\Rightarrow \frac{Y[e^{jw}]}{X[e^{jw}]} = H[e^{jw}] = \left[\frac{1}{1-\frac{3}{4}e^{-jw}+\frac{1}{8}e^{-2jw}}\right]$$

$$\Rightarrow H[e^{jw}] = \frac{1}{1-\frac{1}{2}e^{-jw}-\frac{1}{4}e^{-jw}+\frac{1}{8}e^{-2jw}}$$

$$\Rightarrow H[e^{jw}] = \frac{1}{\left[1-\frac{1}{2}e^{-jw}\right]-\frac{1}{4}e^{-jw}\left[1-\frac{1}{2}e^{-jw}\right]}$$

$$\Rightarrow H[e^{jw}] = \left[\frac{1}{\left[1-\frac{1}{4}e^{-jw}\right]\left[1-\frac{1}{2}e^{-jw}\right]}\right]$$

$$\Rightarrow H[e^{jw}] = \left[\frac{A}{1-\frac{1}{4}e^{-jw}} + \frac{B}{1-\frac{1}{2}e^{-jw}}\right]$$

$$A = \left[\frac{1}{1-\frac{1}{2}e^{-jw}}\right] = \left[\frac{1}{1-\frac{1}{2}(4)}\right]$$

$$= \frac{1}{1-2} = -1$$

$$B = \frac{1}{1-\frac{1}{4}e^{-jw}} = \left[\frac{1}{1-\frac{1}{4}(2)}\right]$$

$$= \frac{1}{1-\frac{1}{2}} = 2$$

Now 9

$$H[e^{jw}] = \left[\frac{-1}{1-\frac{1}{4}e^{-jw}} + \frac{2}{1-\frac{1}{2}e^{-jw}}\right]$$

$$\Rightarrow h[n] = \left[-\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n]\right]$$

$$\Rightarrow \boxed{h[n] = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right] u[n]}$$

Since $x(n) = \delta(n)$

then the output of the LTI system is the response (impulse) of the system

So

$$y(n) = h(n)$$

$$\Rightarrow y(n) = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right] u(n)$$

Again

$$H[e^{jw}] = \frac{1}{1-\frac{3}{4}e^{-jw} + \frac{1}{8}e^{-2jw}}$$

$$= \left[\frac{8}{8-6e^{-jw}+1e^{-2jw}}\right]$$

$$= \frac{8}{8-6[\cos w - j\sin w] + [\cos 2w - j\sin 2w]}$$

10

$$\Rightarrow H[e^{jw}] = \left[\frac{8}{(8-6\cos w+\cos 2w)+j(6\sin w-\sin 2w)}\right]$$

$$\Rightarrow H[e^{jw}] = \left[\frac{8[(8-6\cos w+\cos 2w)-j(6\sin w-\sin 2w)]}{(8-6\cos w+\cos 2w)^2+(6\sin w-\sin 2w)^2}\right]$$

Magnitude Response is given as

$$|H[e^{jw}]| = \frac{8}{u}\left[\{8-6\cos w+\cos 2w\}^2+\{6\sin w-\sin 2w\}^2\right]^{1/2}$$

where

$$u = \left[(8-6\cos w+\cos w)^2+(6\sin w-\sin 2w)^2\right]^{4}$$

Phase Response is given as

$$\angle H[e^{jw}] = \left[\tan^{-1}\left|\frac{\sin 2w-6\sin w}{8-6\cos w+\cos 2w}\right|\right]$$